

BRITISH MATHEMATICAL OLYMPIAD

Round 1 : Wednesday, 13 January 1999

Time allowed *Three and a half hours.*

Instructions • *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*

- *One complete solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top left hand corner.*

Do not turn over until **told to do so**.

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1. I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest will be equal to the sum of the squares of the other two children.

Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

2. A circle has diameter AB and X is a fixed point of AB lying between A and B . A point P , distinct from A and B , lies on the circumference of the circle. Prove that, for all possible positions of P ,

$$\frac{\tan \angle APX}{\tan \angle PAX}$$

remains constant.

3. Determine a positive constant c such that the equation

$$xy^2 - y^2 - x + y = c$$

has precisely three solutions (x, y) in positive integers.

4. Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example,

$$98 = (1 \times 81) + (0 \times 27) + (1 \times 9) + (2 \times 3) + (2 \times 1) = (10122)_3.$$

Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m ; thus, for instance

$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$

Let n be any fixed positive integer. Define the sequence (u_r) by

$$u_1 = n \quad \text{and} \quad u_r = c(u_{r-1}) \quad \text{for} \quad r \geq 2.$$

Show that there is a positive integer r for which $u_r = 1, 2$ or 17 .

5. Consider all functions f from the positive integers to the positive integers such that

(i) for each positive integer m , there is a unique positive integer n such that $f(n) = m$;

(ii) for each positive integer n , we have

$$f(n+1) \text{ is either } 4f(n) - 1 \text{ or } f(n) - 1.$$

Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).